Mass Quantization of the Schwarzschild Black Hole

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Abstract

We examine the Wheeler-DeWitt equation for a static, eternal Schwarzschild black hole in Kuchař-Brown variables and obtain its energy eigenstates. Consistent solutions vanish in the exterior of the Kruskal manifold and are non-vanishing only in the interior. The system is reminiscent of a particle in a box. States of definite parity avoid the singular geometry by vanishing at the origin. These definite parity states admit a discrete energy spectrum, depending on one quantum number which determines the Arnowitt-Deser-Misner (ADM) mass of the black hole according to a relation conjectured long ago by Bekenstein, $M \sim \sqrt{n} M_p$. If attention is restricted only to these quantized energy states, a black hole is described not only by its mass but also by its parity. States of indefinite parity do not admit a quantized mass spectrum.

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General dimensional arguments seem to suggest that true quantum gravity effects should become manifest only at the Planck scale, $M_p \sim 10^{19}$ Gev, $l_p \sim 10^{-35}$ m. Probing such distance scales directly is well beyond the realm of experimental possibility, but one may well ask whether or not there are non-perturbative effects that manifest themselves at more reasonable energy scales, or equivalently at distance scales more accessible to the laboratory.

Where would events occur which could have observational implications? Probably, the best candidates would involve a system of collapsing matter at the very end stages of collapse. According to classical general relativity, a very massive star will eventually undergo continuous collapse until it finally forms a singularity. The precise nature of the singularity, for example, whether it is covered or naked, is not known,[1] but there is general agreement that naked singularities are pathological and the end state will be a covered singularity or a black hole. This expectation has led to a remarkable amount of research over the past several decades on the physics of black holes which, in turn, has resulted in the unfolding of a long list of interesting properties. Perhaps more importantly, black hole physics has forced us to take a harder look at some very fundamental, and yet unanswered, questions of principle. Thus, (a) is the Hawking radiation[2] truly thermal and, if so, why? (b) why is the entropy of a black hole proportional to its area?[3,4,5] (c) what is the end state of an evaporating black hole? (d) what happens to all the information that disappeared down the hole? All of these questions are related and any attempt to answer them satisfactorily must involve a full and consistent quantization of the black hole geometry.

In this letter we apply midi-superspace techniques[6] to study the quantum states of an eternal black hole. We conclude that a normalizable wave function must vanish everywhere in the exterior of the Kruskal manifold, having support only in its interior. They are not *a priori* required to have definite parity, but the definite parity eigenstates depend on one quantum number and the system is reminiscent of a simple particle in a box. States of definite parity do not support the singular geometry and, if attention is restricted to them, then a black hole must be defined

both by its mass and parity. The single quantum number determines the energy of the black hole, so the ADM mass is quantized in units of the Planck mass according to a formula conjectured and employed by Bekenstein,[4,7] $M \sim \sqrt{n} M_p$. The result implies that Hawking radiation is *not* thermal and has been used to derive the area law of black hole thermodynamics.[4] In what follows, we will take $\hbar = c = G = 1$.

We begin with the general problem of the collapse of pressureless dust, described by an action of the form

$$S \;=\; S_g + S_m \;=\; -\frac{1}{16\pi}\int d^4x\sqrt{-g}\mathcal{R} \;-\; \int d^4x\sqrt{-g}\epsilon(x)\left[g_{\alpha\beta}U^\alpha U^\beta + 1\right] \quad (1)$$

where $\mathcal{R}$ is the scalar curvature, $\epsilon(x)$ is the density of the collapsing matter in its proper frame and U^α is the dust velocity. Although the above action describes the collapse of inhomogeneous, pressureless dust in general, we will be concerned with the eternal Schwarzschild black hole in this letter. To recover the Schwarzschild vacuum, we will require (at a later stage) that the Schwarzschild mass function, M, be constant. The dust will then become tenuous, playing only the role of a timekeeper according to a scheme first introduced by Brown and Kuchař.[8]

The action (1) can be cast in canonical form by employing the usual ADM[9] 3+1 dimensional split of a spherically symmetric spacetime

$$ds^2 \;=\; N^2dt^2 \;-\; L^2(dr - N^r dt)^2 \;-\; R^2 d\Omega^2, \quad (2)$$

where $N = N(t,r)$ and $N^r = N^r(t,r)$ are the lapse and shift functions, $\epsilon = \epsilon(t,r)$, $L = L(t,r)$, and $R = R(t,r)$ is the physical radius or curvature coordinate. As emphasized by Regge and Teitelboim[10], it is important to make no specific gauge choices at this stage of the problem; otherwise, essential elements of the canonical structure are eliminated. Using (2), the gravitational part of the action together with the dust can be cast into the form

$$\begin{aligned} S \;=\; &\int dtdr\left[P_L\dot{L} \;+\; P_R\dot{R} \;+\; P_\tau\dot{\tau} \;-\; NH \;-\; N^rH_r\right] \\ &+\; surface\ terms, \end{aligned} \quad (3)$$

where we have introduced, following Brown and Kuchař,[8] the dust proper time

variable, τ, which in general will serve as extrinsic time. P_L and P_R are the momenta conjugate to L and R respectively, and the super hamiltonian, H, and super momentum, H_r, are respectively given by

$$H \;=\; -\left[\frac{P_L P_R}{R} \;-\; \frac{L P_L^2}{2R^2}\right] \;+\; \left[-\,\frac{L}{2} \;-\; \frac{B'^2}{2L} \;+\; \left(\frac{BB'}{L}\right)'\right] \;+\; P_\tau\sqrt{1 \;+\tau'^2/L^2} \tag{4}$$

and

$$H_r \;=\; R'P_R \;-\; LP_L' \;+\; \tau' P_\tau, \tag{5}$$

where the prime denotes a derivative with respect to the ADM label coordinate r. The constraints in the above form do not "decouple" and are very difficult to resolve as they stand. From the general system in (2), Kuchař[11] showed how one can pass by a canonical transformation to a new canonical chart with coordinates M and R together with their conjugate momenta, P_M and $\overline{P}_R$, where M is the Schwarzschild "mass" and R is the curvature coordinate. In this system the constraints are greatly simplified and the phase space variables have immediate physical significance. The canonical transformation is well-defined as long as the metric obeys standard fall-off conditions[11] and, as long as these fall-off conditions are obeyed, the surface action can be recast in the form

$$surface\ terms \;=\; \int dt\,[\pi_+\dot{\tau}_+ \;+\; \pi_-\dot{\tau}_- \;-\; N_+C_+ \;-\; N_-C_-]\,, \tag{6}$$

where $\tau_\pm$ are the proper times measured on the parametrization clocks at right (left) infinity. The constraints $C_\pm = \pm\pi_\pm + M_\pm$ identify their conjugate momenta as the mass at right (left) infinity. In terms of the new variables the entire action, along with the surface term is

$$S \;=\; \int dt \int dr\left[\overline{P}_M\dot{M} \;+\; \overline{P}_R\dot{R} \;+\; \overline{P}_\tau\dot{\tau} \;-\; NH \;-\; N^r H_r\right], \tag{7}$$

where $\overline{P}_M = P_M - \tau'$, $\overline{P}_\tau = P_\tau + M'$ and P_M and P_τ are the original Brown-Kuchař variables. The transformations leading to these variables may be found in

refs.[8,11]. In passing to the transformed momenta, $\overline{P}_\tau$ and $\overline{P}_M$, we have made a canonical transformation generated by $M\tau'$, which effectively absorbs the surface terms. (We have implicitly fixed the dust proper time to coincide at infinity with the parametrization clocks.)

The super hamiltonian and super momentum constraints become

$$\begin{aligned} H &= -\left[\frac{F^{-1}M'R' + F\overline{P}_R(\overline{P}_M + \tau')}{L}\right] \\ &\quad + \ (\overline{P}_\tau - M')\sqrt{1+\tau'^2/L^2} \ = \ 0 \end{aligned} \tag{8}$$

and

$$H_r \ = \ M'\overline{P}_M \ + \ R'\overline{P}_R \ + \ \tau'\overline{P}_\tau \ = \ 0, \tag{9}$$

where we have used

$$L^2 \ = \ F^{-1}R'^2 \ - \ F(\overline{P}_M + \tau')^2 \tag{10}$$

and $F \ = \ 1 - 2M/R$. L^2, being the component g_{rr} of the spherically symmetric metric in (2), must be positive definite everywhere. F is positive in the exterior (Schwarzschild) region and negative in the interior and this will play an important role in the consistency conditions that follow. By direct computation of Poisson brackets, it is easy to determine the "velocities" in terms of the conjugate momenta from the above expressions and they are

$$\begin{aligned} \dot{\tau} &= N\sqrt{1+\tau'^2/L^2} \ + \ N^r\tau', \\ \dot{R} &= -\frac{NF(\overline{P}_M + \tau')}{L} \ + \ N^r R', \\ \dot{M} &= \frac{NR'\tau'(\overline{P}_\tau - M')}{L^3} \ + \ N^r M'. \end{aligned} \tag{11}$$

To proceed with the quantization program, one must raise the canonical variables to operator status and consider the constraints in (7) as operator constraints on

the wave functional $\Psi(\tau, R, M)$, *i.e.,* one sets

$$\hat{H}\Psi \;=\; 0 \;=\; \hat{H}_r\Psi. \tag{12}$$

The second constraint enforces spatial diffeomorphism invariance of the wave functional on hypersurfaces orthogonal to the dust proper time. It simply says that $\Psi' = 0$, where the derivative is with respect to the label coordinate r. It is convenient to use the super momentum constraint in (9) to eliminate $\overline{P}_M$ in the expression for the super hamiltonian in (8). The super hamiltonian constraint turns into

$$(\overline{P}_\tau - M')^2 \;+\; F\overline{P}_R^2 \;-\; \frac{M'^2}{F} \;=\; 0. \tag{13}$$

We will now specialize to the black hole by requiring $M' = 0$ so that only the homogeneous mode of $M(t, r)$ survives. The resulting equation decouples and is hyperbolic in the region $R < 2M$ (the interior of the Kruskal manifold) but elliptic in the region $R > 2M$ (the exterior). This is because the quantity F is negative in the interior, but positive in the exterior – so the "kinetic energy" there has the "wrong" sign. The two distinct solutions must agree on the boundary. States of the quantum theory are described by functions of the configuration space variables, τ, R and M.

To obtain the wave equation (12), we replace $\overline{P}_\tau = i\nabla_\tau$ and $\overline{P}_R = -i\nabla_R$ in (13) with the result

$$\nabla^2\Psi \;=\; G^{ab}\nabla_a\nabla_b\Psi \;=\; \tilde{H}\Psi \;=\; 0, \tag{14}$$

where G_{ab} is the field space metric, $G_{ab} = \mathrm{diag}(1, 1/F)$ and ∇_a is the covariant derivative with respect to this metric. Eq. (14) is a massless "Klein-Gordon" like equation. We will consider only its positive energy solutions, $\Psi(\tau, R, M) = e^{-iE\tau}\psi(R, M)$, beginning with the exterior.

The configuration space metric, $G_{ab} = \mathrm{diag}(1, 1/F)$ $(F > 0)$, defines a natural measure on the Hilbert space. Because it is flat, it is convenient to transform to the coordinate

$$R_* \;=\; \int dR\sqrt{\frac{R}{R-2M}} \;=\; \sqrt{R(R-2M)} + M\ln[R-M+\sqrt{R(R-2M)}] \quad (15)$$

and use R_* instead of the original curvature coordinate, R, above. Clearly, $R_* \,\epsilon\, (M\ln M, \infty)$ and the wave equation, $\partial_\tau^2\Psi = -\partial_*^2\Psi$, describes the quantum theory whose Hilbert space is $\mathcal{H} := \mathcal{L}^2(\mathbf{R}, dR_*)$ with inner product

$$\langle\Psi_1, \Psi_2\rangle \;=\; \int_{M\ln M}^{\infty} dR_*\Psi_1^\dagger\Psi_2 \;. \quad (16)$$

The positive energy solution that is well behaved in the entire range of R_* has the form

$$\Psi_{out}(R, M, \tau) \;=\; b(M)e^{-iE(\tau - iR_*)}, \quad (17)$$

where $b(M)$ is an arbitrary function on M. We will now argue thatΨ_{out} is identically zero. Spatial diffeomorphism invariance of Ψ_{out} on the hypersurface orthogonal to τ implies that $E(\tau' - iR_*')\Psi = 0$. This is met by $\tau' = 0 = R'$, by $E = 0$, or by $b(M) = 0$. The condition $\tau' = 0 = R'$ is unacceptable because the positivity of L^2 precludes $R' = 0$ in the exterior, though τ' can be vanishing. One could take $\Psi = b(M)$ as a consistent exterior solution and this was originally proposed by Kuchař[11]. Then $E = 0$ and M is automatically constant from (11) but, because M is a constant and the field space is not compact in this region, this solution would not be normalizable. Furthermore, $E = 0$ would represent an uninteresting zero total energy solution, so we take $b(M) = 0$ in the exterior, *i.e.*, $\Psi_{out} = 0$.

Next let us consider the solution in the internal region, $R < 2M$ $(F < 0)$. Here the equation is hyperbolic and it is convenient to transform to the coordinate $\overline{R}_*$

defined by

$$\overline{R}_* \;=\; -\sqrt{R(2M-R)} \;+\; M\tan^{-1}\left[\frac{R-M}{\sqrt{R(2M-R)}}\right] \;. \tag{18}$$

The new coordinate lies in the range $(-\frac{\pi M}{2}, +\frac{\pi M}{2})$ and the wave equation, $\partial_\tau^2\Psi = \partial_*^2\Psi$ now defines the quantum theory whose Hilbert space is $\mathcal{H} := \mathcal{L}^2(\mathbf{R}, d\overline{R}_*)$ with inner product

$$\langle\Psi_1, \Psi_2\rangle \;=\; \int_{-\frac{\pi M}{2}}^{+\frac{\pi M}{2}} d\overline{R}_* \Psi_1^\dagger \Psi_2 \;. \tag{19}$$

The general (positive energy) solution is

$$\Psi_{in} \;=\; c_+(M)e^{-iE(\tau+\overline{R}_*)} \;+\; c_-(M)e^{-iE(\tau-\overline{R}_*)} \tag{20}$$

where $c_\pm$ are functions only of M. Again we must impose the super momentum constraint, which reads

$$(\tau' + R')c_+(M)e^{-iE(\tau+\overline{R}_*)} \;+\; (\tau' - R')c_-(M)e^{-iE(\tau-\overline{R}_*)} \;=\; 0, \tag{21}$$

assuming $E > 0$. A consistent and physically meaningful solution to this equation is $\tau' = \overline{R}_*' = 0$. Returning to (11), we see that the choice implies that $\dot{\tau} = N$ and $\dot{M} = 0$. Setting $N = 1$, the dust proper time turns into the asymptotic Minkowski time and the energy, E, should be associated with the ADM mass of the black hole.

This solution must match the solution in the exterior at $R = 2M$. Matching gives

$$c_-(M) \;=\; -\,c_+(M)e^{-iEM\pi}, \tag{22}$$

so that

$$\Psi_{in} \;=\; c_+(M)\left[e^{-iE(\tau+\overline{R}_*)} \;-\; e^{-iEM\pi}e^{-iE(\tau-\overline{R}_*)}\right] \;. \tag{23}$$

Because the parity operator, $\overline{R}_* \to -\overline{R}_*$, commutes with the "Hamiltonian" operator, $\tilde{H}$, states of definite parity are guaranteed to remain so for all times. The

definite parity eigenstates, apart from obeying the symmetries of $\tilde{H}$ and the domain of $\overline{R}_*$, vanish at $R = 0$ ($\overline{R}_* = -\frac{\pi M}{2}$). Therefore they provide no support for the classical singular geometry. These definite parity eigenstates exhibit a discrete energy spectrum and are given by

$$\begin{aligned} \Psi_{in}^{(+)} &= \frac{1}{\sqrt{\pi M}} e^{-iE\tau} \cos E\overline{R}_* \qquad & EM &= (2n+1) \ , \\ \Psi_{in}^{(-)} &= \frac{1}{\sqrt{\pi M}} e^{-iE\tau} \sin E\overline{R}_* \qquad & EM &= 2n \ , \end{aligned} \tag{24}$$

where we take $n \,\epsilon\, \mathbf{N} \cup \{0\}$ to maintain the positivity of the total energy, that is, to agree with the classical positive energy theorems. If we identify the total energy with the ADM mass of the black hole, then the ADM mass is quantized in units of the Planck mass,

$$M = \sqrt{n} M_p, \qquad\qquad n \,\epsilon\, \mathbf{N} \tag{25}$$

as proposed in the introduction. Restricting attention to states of definite parity seems very natural for the reasons mentioned above. It raises an intriguing possibility: that quantum black holes are determined by their parity as well as by their mass.

Considerations involving the quantization of the angular momentum and charge of extremal and non-extremal Reissner-Nordstrom and Kerr-Newman black holes led Bekenstein to propose, long ago, that the horizon area of a black hole was quantized in integer multiples of a fundamental area, presumably of the order of l_p^2. Because the area of the horizon is proportional to the square of the mass of a black hole, Bekenstein's original proposal can be viewed as a proposal for the spectrum of the (quantum) mass operator and coincides exactly with our formula (24). The consequences of this mass spectrum have been discussed extensively by Bekenstein and Mukhanov.[12] Many attempts, ranging from quantum membrane and string theory approaches[13], to canonical quantum gravity treatments[14] of collapsing matter and vacuum spacetimes, have been made to obtain Bekenstein's

formula from first principles. These attempts have had varying results which depend strongly on the simplifying assumptions made either in the model itself or in the steps leading to the Hamiltonian reduction. Some obtain equally spaced area levels and others do not.

We have used the Hamiltonian reduction of spherical geometries by Kuchař[11] and the coupling to incoherent dust by Brown and Kuchař[8] to quantize the Schwarzschild black hole. We have only required homogeneity and the positivity of the black hole mass. Thus the dust was made tenuous and clocks attached to the dust particles served to identify the time foliation. Time evolution appeared naturally in the formalism. An appropriate choice of the lapse function fixed the dust proper time to coincide with the proper time of a static asymptotic observer and the total energy was identified with the ADM mass of the hole. A quantum black hole behaves very much like a particle in a box. The wave function vanishes in the exterior, therefore the dynamics of the system takes place in the interior of the Kruskal manifold. This reflects the fact that the exterior region is a vacuum for the asymptotic observer. In the interior, normalized solutions of definite parity are quantized in half-integer units. We know of no reason to select only states of definite parity, other than the fact that parity is a discrete symmetry of the system and that these states exhibit a node at the classical singularity. This means that they do not support the singular geometry.

In this letter we have confined our study to the case of the eternal Schwarzschild black hole though we foresee no obstacle to treating charged and rotating black holes. Eq. (13) describes the general problem of the collapse of incoherent dust. Classically, models of pressureless dust collapse lead both to covered and naked singularities. Semi-classical considerations indicate that the Hawking radiation is vastly different in the two cases.[15] The natural next step is to examine Hawking radiation in this context, but additional degrees of freedom that carry the Hawking radiation must then be introduced.

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